Growth of leveraged ETF

Denote:

x_i – daily return on the i –th trading day

n – number of trading days

p – value of the index (or unleveraged ETF) after n days, if its value on day 0 is 1. Obviously,

$$p = \prod_{i=1}^{n} (1 + x_i)$$

Let's find V - the value of triple-leveraged ETF (whose return on day i is $3x_i$) after n days. Again, value on day 0 is 1.

$$V = \prod_{i=1}^{n} (1 + 3x_i) = \prod_{i=1}^{n} [(1 + x_i)^3 - 3{x_i}^2 - {x_i}^3]$$

Denote:

$$\alpha_i = 3{x_i}^2(1 + \frac{x_i}{3})$$

Consequently, and keeping in mind that $p^3 = \prod_{i=1}^{n} \left(1 + x_j\right)^3$,

$$V = \prod_{i=1}^{n} [(1 + x_i)^3 - \alpha_i] \approx \prod_{i=1}^{n} (1 + x_i)^3 - \sum_{i=1}^{n} \alpha_i \prod_{i \neq j} \left(1 + x_j\right)^3 =$$

$$= \prod_{i=1}^{n} (1 + x_i)^3 - \sum_{i=1}^{n} \frac{\alpha_i}{(1 + x_i)^3} \prod_{j=1}^{n} \left(1 + x_j\right)^3 =$$

$$= p^3 - p^3 \sum_{i=1}^{n} \frac{\alpha_i}{(1 + x_i)^3} = p^3 - 3p^3 \sum_{i=1}^{n} {x_i}^2 [1 - \frac{1 + \frac{x_i}{3}}{(1 + x_i)^3}]$$

$$=$$

In the previous formula we took into account that ${x_i}^2$«1, and thus omitted the members of 4th degree and higher for x_i (i.e. 2nd degree and higher for α_i).

If, additionally, we assume that x_i«1 (small daily changes), we can further simplify the last expression:

$$V \approx p^3 - 3p^3 \sum_{i=1}^{n} {x_i}^2 = p^3(1 - 3nX^2)$$

Here X is the RMS (root-mean-square) of daily returns during the period of n trading days.

Conclusions:

1. Whenever the market moves predominantly in one direction, performance of the 3xETF is almost equal to the cube of the underlying fund performance, but is always slightly less. Say, if the index doubles, 3xETF grows almost 8 times; if halves – drops to less than 12.5% of the initial value.
2. Second member, $-3p^3nX^2$, is always negative, always claws back from the leveraged ETF performance, wherever the index moves – that is the component responsible for erosion. It is more sensitive to daily changes (proportionate to their square) than to length of the period, n. That means that the erosion for index ETFs would be very moderate, while for leveraged resource funds or leveraged VIX (big daily swings) it may be significant. Example: For a normal average year for S&P, growth 8% (i.e. p=1.08), n=250, X=0.5% (which may be an exaggeration), we find: $p^3 = 1.26, 3npX^2 = 0.024, V = 1.236$. That means that erosion is only 2.4%, leaving the performance at 23.6%, so erosion is not a major factor.
3. Erosion becomes decisive when the index returns to the same point after some time. In that case, $p = 1$, zero growth of the index, and performance of the leveraged ETF is negative, equal to the erosion component. That may be huge at comebacks after bear markets (significant time and higher average RMS daily return).
4. Both positive and negative days contribute equally to the erosion component (all enter the RMS as squares). Volatility as such doesn't contribute to erosion, it rather diminishes the growth (i.e. p), and hence makes the erosion factor

more visible. Also high volatility usually means larger daily swings, increasing X and hence – erosion.

5. It is quite obvious that for 2x leveraged ETFs, the formula modifies like this:

$$V \approx p^3(1 - 2nX^2)$$

Squared dependence – still not bad for strong bull markets.